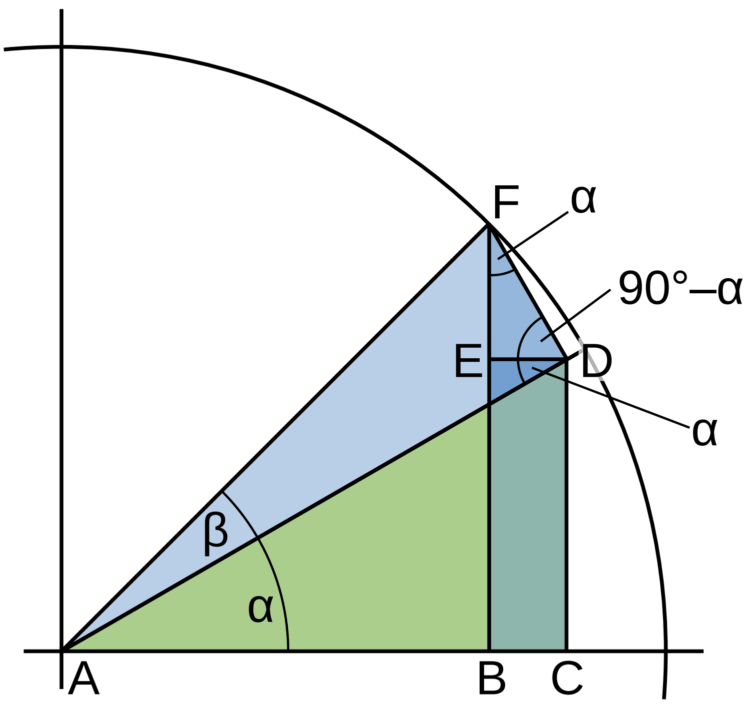
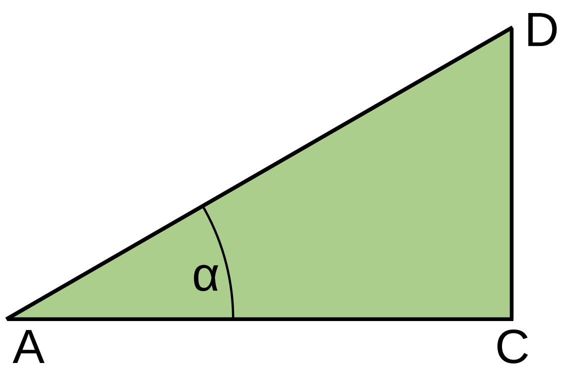
Ableitung Sinus

Trigonometrische Additionstheoreme

Behauptung: $\sin(\alpha+\beta)=\sin(\alpha)\cdot\cos(\beta)+\cos(\alpha)\cdot\sin(\beta)$

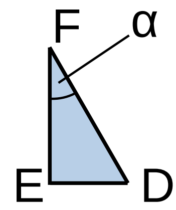
Beweis:





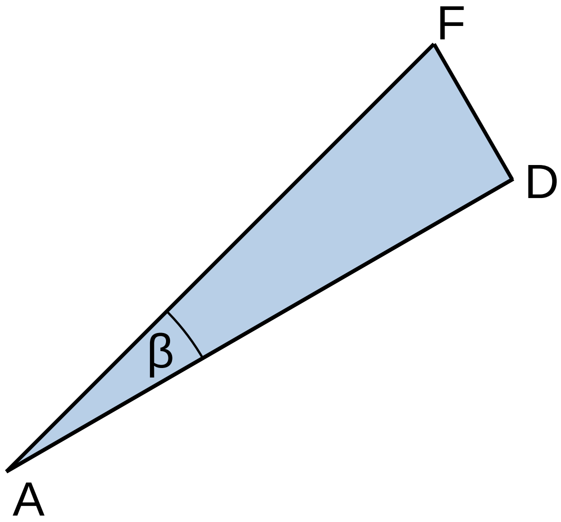
$$\sin(\alpha)=\frac{\overline{CD}}{\overline{AD}} \Rightarrow \overline{CD}=\sin(\alpha)\cdot\overline{AD}$$

$$\cos(\alpha)=\frac{\overline{AC}}{\overline{AD}}$$



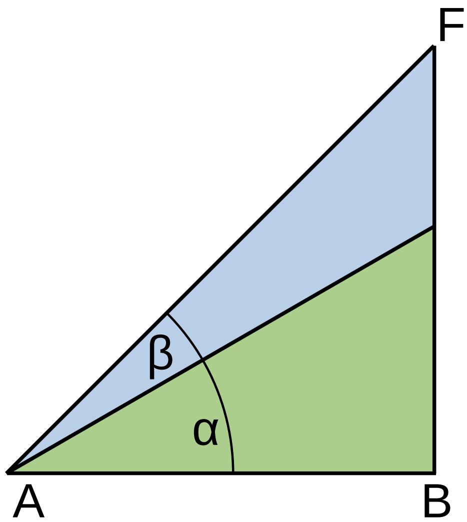
$$\sin(\alpha)=\frac{\overline{DE}}{\overline{DF}}$$

$$\cos(\alpha)=\frac{\overline{EF}}{\overline{DF}} \Rightarrow \overline{EF}=\cos(\alpha)\cdot\overline{DF}$$



$$\sin(\beta)=\frac{\overline{DF}}{\overline{AF}} \underset{\overline{AF}=1}{=} \overline{DF}$$

$$\cos(\beta)=\frac{\overline{AD}}{\overline{AF}} \underset{\overline{AF}=1}{=} \overline{AD}$$



$$\sin(\alpha+\beta)=\frac{\overline{BF}}{\overline{AF}} \underset{\overline{AF}=1}{=} \overline{BF}$$

$$\cos(\alpha+\beta)=\frac{\overline{AB}}{\overline{AF}} \underset{\overline{AF}=1}{=} \overline{AB}$$

Mit $\overline{BF}=\overline{CD}+\overline{EF}$ und $\overline{AC}=\overline{AC}-\overline{BC}$ folgt:

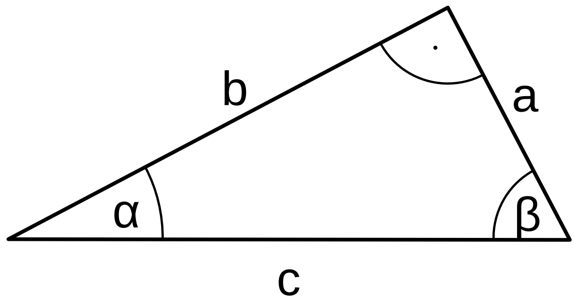
$$\sin(\alpha+\beta)=\overline{BF}=\overline{CD}+\overline{EF}=\sin(\alpha)\cdot\overline{AD}+\cos(\alpha)\cdot\overline{DF}=\sin(\alpha)\cdot\cos(\beta)+\cos(\alpha)\cdot\sin(\beta)$$

Behauptung: $\sin(2\alpha)=2\cdot\sin(\alpha)\cdot\cos(\alpha)$

Beweis: $\sin(2\alpha)=\sin(\alpha+\alpha)=\sin(\alpha)\cdot\cos(\alpha)+\cos(\alpha)\cdot\sin(\alpha)=2\cdot\sin(\alpha)\cdot\cos(\alpha)$

Behauptung: $\sin^2(\alpha)+\cos^2(\alpha)=1$

Beweis:



$$\sin(\alpha)=\frac{a}{c} \;\Rightarrow\; a=c\cdot\sin(\alpha)$$
$$\cos(\alpha)=\frac{b}{c} \;\Rightarrow\; b=c\cdot\cos(\alpha)$$

nach dem Satz von Pythagoras gilt:

$$\begin{aligned} a^2+b^2=c^2 &\Leftrightarrow \left(c\cdot\sin(\alpha)\right)^2+\left(c\cdot\cos(\alpha)\right)^2=c^2\cdot\sin^2(\alpha)+c^2\cdot\cos^2(\alpha)=c^2\left(\sin^2(\alpha)+\cos^2(\alpha)\right)=c^2 \\ &\Leftrightarrow \sin^2(\alpha)+\cos^2(\alpha)=1 \end{aligned}$$

Behauptung: $\sin(\alpha)-\sin(\beta)=2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$

Beweis:

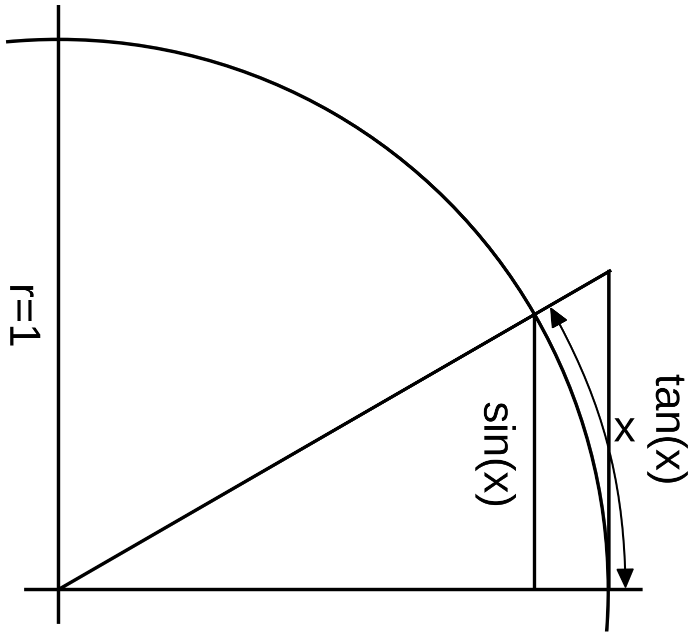
$$\begin{aligned} & 2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right) \\ &=2\cdot\left[\cos\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)-\sin\left(\frac{\alpha}{2}\right)\cdot\sin\left(\frac{\beta}{2}\right)\right]\cdot\left[\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)-\cos\left(\frac{\alpha}{2}\right)\cdot\sin\left(\frac{\beta}{2}\right)\right] \\ &=2\cdot\left[\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)\cdot\cos^2\left(\frac{\beta}{2}\right)-\sin\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)\right. \\ &\qquad \left.-\sin^2\left(\frac{\alpha}{2}\right)\cdot\sin\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)+\sin\left(\frac{\alpha}{2}\right)\cdot\sin^2\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)\right] \\ &=2\cdot\left[\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)\cdot\cos^2\left(\frac{\beta}{2}\right)+\sin\left(\frac{\alpha}{2}\right)\cdot\sin^2\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)\right. \\ &\qquad \left.-\sin^2\left(\frac{\alpha}{2}\right)\cdot\sin\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)-\sin\left(\frac{\beta}{2}\right)\cdot\cos^2\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)\right] \\ &=2\cdot\left[\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)\cdot\left(\cos^2\left(\frac{\beta}{2}\right)+\sin^2\left(\frac{\beta}{2}\right)\right)\right. \\ &\qquad \left.-\left[\sin\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)\cdot\left(\sin^2\left(\frac{\alpha}{2}\right)+\cos^2\left(\frac{\alpha}{2}\right)\right)\right]\right] \\ &=2\cdot\left[\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)-\sin\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)\right] \end{aligned}$$

$$=2\cdot\sin\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{2}\right)-2\cdot\sin\left(\frac{\beta}{2}\right)\cdot\cos\left(\frac{\beta}{2}\right)$$
$$=\sin(\alpha)-\sin(\beta)$$

Benötigter Grenzwert

Behauptung: $\lim\limits_{x\to 0}\left(\frac{\sin(x)}{x}\right)=1$

Beweis:



Das Schaubild zeigt, dass für $0\leq x<\frac{\pi}{2}$ folgendes gilt:

$$\begin{array}{rcccl|l}
\sin(x) & \leq & x & \leq & \tan(x) & \tan \text{ ersetzen} \\
\sin(x) & \leq & x & \leq & \frac{\sin(x)}{\cos(x)} & \div\sin(x) \\
1 & \leq & \frac{x}{\sin(x)} & \leq & \frac{1}{\cos(x)} & \text{Kehrwert bilden} \\
1 & \geq & \frac{\sin(x)}{x} & \geq & \cos(x) & \text{Grenzwerte bilden} \\
1=\lim\limits_{x\to 0}(1) & \geq & \lim\limits_{x\to 0}\left(\frac{\sin(x)}{x}\right) & \geq & \lim\limits_{x\to 0}(\cos(x))=1 &
\end{array}$$

damit muss $\lim\limits_{x\to 0}\left(\frac{\sin(x)}{x}\right)=1$ sein.

Ableitung vom Sinus

Behauptung: $(\sin(x))' = \cos(x)$

Beweis:

$$\lim_{h \to 0} \left(\frac{\sin(x+h) - \sin(x)}{h} \right)$$

Substitution: $x+h \to \alpha \ \wedge \ x \to \beta$

$$\lim_{h \to 0} \frac{\sin(\alpha) - \sin(\beta)}{h}$$

$$=_{\text{Additionstheorem}} \lim_{h \to 0} \frac{2 \cdot \cos\left(\frac{\alpha+\beta}{2}\right) \cdot \sin\left(\frac{\alpha-\beta}{2}\right)}{h}$$

Rücksubtitution: $\alpha \to x+h \ \wedge \ \beta \to x$

$$\lim_{h \to 0} \left(\frac{2 \cdot \cos\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x+h-x}{2}\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{2 \cdot \cos\left(x + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\cos\left(x + \frac{h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right)$$

$$= \lim_{h \to 0} \left(\underbrace{\cos\left(x + \frac{h}{2}\right)}_{\to \cos(x)} \cdot \underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{\to 1} \right)$$

$$= \cos(x)$$